For this configuration and $\epsilon_r = 4$, we have obtained

$$Z_{0o} = 47.75\,\Omega, \qquad Z_{0e} = 127.2\,\Omega.$$

To have an idea of the precision of the potential calculation, we have calculated the capacitance per unit length of the stripline by application of Gauss' theorem for two surfaces. The first, near the strip, gives C_I, and the second, near the external walls of the box, gives C_0. We call relative precision of the calculation the ratio $[C_I - C_0]/C_I$.

Table I gives data for a simple line when we stop the calculation if the highest difference between the potentials of the corresponding nodes for two successive iterations is smaller than a fixed value, called "test$_1$."

Table II gives data for a microstrip coupler. Here, C_I is the capacitance calculated for a surface surrounding one of the two strips, for instance, the positively charged one in the odd mode. C_0 is calculated for a surface containing all the other conductors.

Table III gives results for the same number of iterations (36) in the case of the coupler with $\epsilon_r = 4$.

Conversely, in Table IV, we give the results when we stop the computation if the "relative precision" is smaller than a fixed value called "test$_2$."

It should be noted that for very small differences (a few percent) in the value of ω, the number of iterations and the precision are perceptibly different. This effect has been observed and justified by some authors [20]–[24].

In Table V, as an illustration of our method against the Gauss–Seidel one, we give the data as for Table IV, with the best accelerating factor and without the accelerating factor.

IV. Conclusion

From these results, it can be seen that the approximations made for the calculation of an accelerating factor are very good. In each case we have tested, we have obtained an important amelioration either in the computational time or in precision of calculation, often for both.

The precision of the finite-differences method is sufficiently good, as can be seen by comparison with results given by others.

We have used two types of results. The first ones are those obtained by Cohn's formulas [11]. In our program, making $\epsilon_r = 1$ and $N \gg M$, L, N_1, we must approach Cohn's case.

For example, we have obtained the following:

M	N	K	L	N_1	Z_{0o} (calculated)	Z_{0e} (calculated)	Z_{0o} (Cohn)	Z_{0e} (Cohn)
21	101	9	5	6	43.3	143.7	45.6	148.8
21	101	9	5	16	17.7	73.6	18.1	74.8
51	101	24	5	6	46.5	239.2	47.9	249.3
51	101	24	5	16	19.7	153	20.1	158.2
53	132	24	7	18	24.7	137.5	25	143.8

We have made 17 comparisons between our results and those calculated using Cohn's formula. The mean accuracy of these results is about 2 percent.

Second, experimental results have been obtained by the Centre National d'Etudes des Télécommunications, Lannion, France. Comparison with these results gives an accuracy of 4 or 5 percent. For example, we have the following:

M	N	K	L	N_1	Z_{0o} (calculated)	Z_{0e} (calculated)	Z_{0o} (experimental)	Z_{0e} (experimental)
18	44	9	2	6	16.15		16.17	
72	88	35	4	19	11	115	10.05	111.3

The only point which can be noted in opposition to this method is that for realistic problems, a computer of great capacity is necessary.

Finally, the finite-differences method appears in many aspects to be the most simple to use for the calculations of microstrip parameters in the TEM approximation.

This work allows a reduction of the computational time necessary in the finite-differences method using the SOR technique of 20–60 percent according to the desired accuracy.

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On Uniform Multimode Transmission Lines

CLAYTON R. PAUL

***Abstract*—In a recent short paper [1], a method for constructing solutions to the classical uniform multiwire transmission-line equations was given which was intended to include the case of partial eigenvalue degeneracy. This development appears to be incorrect and a correct development will be given. In addition, a complete method for constructing the matrix chain parameters of a section of line will be presented.**

We will consider n uniform transmission lines described by the matrix partial differential equations

$$\frac{\partial \boldsymbol{v}(x,t)}{\partial x} = -\boldsymbol{R}\boldsymbol{i}(x,t) - \boldsymbol{L}\frac{\partial \boldsymbol{i}(x,t)}{\partial t} \tag{1a}$$

$$\frac{\partial \boldsymbol{i}(x,t)}{\partial x} = -\boldsymbol{G}\boldsymbol{v}(x,t) - \boldsymbol{C}\frac{\partial \boldsymbol{v}(x,t)}{\partial t} \tag{1b}$$

where $\boldsymbol{v}(x,t)$ and $\boldsymbol{i}(x,t)$ are $n \times 1$ vector functions of the transmission-line voltages with respect to some reference conductor (usually a ground plane) and currents, respectively, as a function of distance x along the line and time t. The matrices $\boldsymbol{R}$, $\boldsymbol{L}$, $\boldsymbol{G}$, and $\boldsymbol{C}$ are $n \times n$ matrices independent of x. Nonuniform transmission lines would have $\boldsymbol{R}$, $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ as functions of x. Usually, $\boldsymbol{R}$ is diagonal and $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ are symmetric (for lines emersed in linear, isotropic media). By invoking the Laplace transform with respect to time, we arrive at the equations

$$\frac{d\boldsymbol{V}(x)}{dx} = -\boldsymbol{Z}\boldsymbol{I}(x) \tag{2a}$$

$$\frac{d\boldsymbol{I}(x)}{dx} = -\boldsymbol{Y}\boldsymbol{V}(x) \tag{2b}$$

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where $\boldsymbol{V}(x)$ and $\boldsymbol{I}(x)$ are the Laplace-transformed vectors $\boldsymbol{v}(x, t)$ and $\boldsymbol{i}(x, t)$, respectively, and $\boldsymbol{Z}=\boldsymbol{R}+p\boldsymbol{L}$ and $\boldsymbol{Y}=\boldsymbol{G}+p\boldsymbol{C}$. Note that the following results hold where we assume sinusoidal excitation of the lines, i.e., $p=j\omega$. $\boldsymbol{L}$, $\boldsymbol{C}$, and $\boldsymbol{G}$ will be independent of p for nondispersive media, and $\boldsymbol{R}$ will usually be a function of p due to skin effect.

Since (2) represents "strongly coupled" differential equations, we may form, by differentiating (2b) with respect to x,

$$\ddot{\boldsymbol{I}}(x) = \boldsymbol{\Gamma}\boldsymbol{I}(x) \tag{3}$$

where $\boldsymbol{\Gamma}=\boldsymbol{YZ}$. The double dot $(\ddot{\cdot})$ notation denotes second derivative with respect to x. Note that $\boldsymbol{Z}$ and $\boldsymbol{Y}$ being symmetric does not insure that $\boldsymbol{\Gamma}$ is symmetric.

We first determine the eigenvalues of $\boldsymbol{\Gamma}$ from

$$\det\left[\gamma^2\boldsymbol{I}_n - \boldsymbol{\Gamma}\right] = \det\left[\gamma^2\boldsymbol{I}_n - \boldsymbol{ZY}\right] = 0 \tag{4}$$

yielding n eigenvalues of $\boldsymbol{\Gamma}$, where $\boldsymbol{I}_n$ is the $n\times n$ identity matrix and $\det[\boldsymbol{M}]$ denotes the determinant of the square matrix $\boldsymbol{M}$. It is well known (see [10] or [3], for example) that there exists an $n\times n$ nonsingular matrix function of the complex variable p, $\boldsymbol{T}$, that transforms $\boldsymbol{\Gamma}$ to the Jordan canonical form

$$\boldsymbol{T}^{-1}\boldsymbol{\Gamma}\boldsymbol{T} = \boldsymbol{\gamma}^2. \tag{5}$$

The $n\times n$ matrix $\boldsymbol{\gamma}^2$ is structured in the following way [5]. If there exist k distinct eigenvalues $\gamma_1^2, \cdots, \gamma_k^2 (k\leq n)$, then $\boldsymbol{\gamma}^2$ will be of block-diagonal form with k blocks $\boldsymbol{\gamma}_i^2$ on the main diagonal of dimension $m_i\times m_i$, where m_i is the multiplicity of γ_i^2, $\Sigma_{i=1}^{k} m_i = n$. Each $\boldsymbol{\gamma}_i^2$ will be block diagonal and will be composed of $\alpha(i)$ separate Jordan blocks on the main diagonal $\boldsymbol{\gamma}_{ij}^2$, all associated with γ_i^2, where $j=1, \cdots, \alpha(i)$. Each $\boldsymbol{\gamma}_{ij}^2$ will be of dimension $n_{ij}\times n_{ij}$ and will have γ_i^2 on the main diagonal, ones on the diagonal immediately above the main diagonal, and zeros elsewhere. It is known also [3] that $\boldsymbol{\gamma}^2$ will be diagonal only if $r[\gamma_i^2\boldsymbol{I}_n - \boldsymbol{\Gamma}] = n - m_i$, for $i=1, \cdots, k$, where we denote the rank of a matrix $\boldsymbol{M}$ over the field of rational polynomials by $r[\boldsymbol{M}]$. If $k=n$, then this is certainly the case, yet diagonalization of $\boldsymbol{\Gamma}$ is not always possible. The diagonalization of $\boldsymbol{\Gamma}$ by a similarity transformation as in (5) is in most cases difficult to determine and is related to the degree of the minimal polynomial of $\boldsymbol{\Gamma}$ [3]. As an example, suppose $\boldsymbol{\Gamma}$ is real, $n=5$, $k=2$, $\gamma_1^2=2$, $\gamma_2^2=3$, $\alpha(1)=1$, $\alpha(2)=2$, $n_{11}=1$, $n_{21}=1$, and $n_{22}=3$. Then $\boldsymbol{\gamma}^2$ may be written as

$$\boldsymbol{\gamma}^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{6}$$

Note that $\boldsymbol{\gamma}^2$ is always unique within a rearrangement of the Jordan blocks on the main diagonal.

At this point, we should note that (5) represents a change of variables $\boldsymbol{I}=\boldsymbol{T}\boldsymbol{I}_m$, where $\boldsymbol{I}_m$ is an $n\times 1$ vector of "modal currents." Thus

$$\ddot{\boldsymbol{I}}_m = \boldsymbol{\gamma}^2\boldsymbol{I}_m. \tag{7}$$

The solution to (7) is easily shown to be

$$\boldsymbol{I}_m = \boldsymbol{S}(x, \gamma)\boldsymbol{e}^{\boldsymbol{\gamma}x}\boldsymbol{a}^{+} + \boldsymbol{S}(x, -\gamma)\boldsymbol{e}^{-\boldsymbol{\gamma}x}\boldsymbol{a}^{-} \tag{8}$$

where $\boldsymbol{e}^{\boldsymbol{\gamma}x}$ is an $n\times n$ diagonal matrix with $e^{\gamma_i x}$ entries on the main diagonal, $i=1, \cdots, k$, with the same sequence as in $\boldsymbol{\gamma}^2$, $\boldsymbol{a}^{+}$ and $\boldsymbol{a}^{-}$ are $n\times 1$ vectors of undetermined constants, and $\gamma_i = \sqrt{\gamma_i^2}$. The $n\times n$ matrix $\boldsymbol{S}(x, \gamma)$ is of block-diagonal form with $\Sigma_{i=1}^{k}\alpha(i)$ blocks $\boldsymbol{S}_{ij}(x, \gamma_i)$ on the main diagonal, each of which is of upper-triangular form with ones on the main diagonal, zeros below the main diagonal, and each block is of dimension $n_{ij}\times n_{ij}$. The elements of $\boldsymbol{S}_{ij}(x, \gamma_i)$ above the main diagonal are linear combinations of powers of x. Each of the blocks $\boldsymbol{S}_{ij}(x, \gamma_i)$ will be associated with the blocks γ_{ij}^2 of the Jordan form $\boldsymbol{\gamma}^2$. Ogata [4] shows the structure for $\boldsymbol{S}(x, \gamma)$, assuming first-order differential equations reduced to Jordan form by a similarity transformation as in (5). For second-order equations reduced to Jordan form, as we have here, a similar development can be made by assuming the given structure for $\boldsymbol{S}_{ij}(x, \gamma_i)$ and substituting (8) into (7). For example, for $n_{ij}=4$, one may show

$$\boldsymbol{S}_{ij}(x, \gamma_i) = \begin{bmatrix} 1 & \frac{x}{2\gamma_i} & \left(\frac{x^2}{8\gamma_i^2} - \frac{x}{8\gamma_i^3}\right) & \left(\frac{x^3}{48\gamma_i^3} - \frac{x^2}{16\gamma_i^4} + \frac{x}{16\gamma_i^5}\right) \\ 0 & 1 & \frac{x}{2\gamma_i} & \left(\frac{x^2}{8\gamma_i^2} - \frac{x}{8\gamma_i^3}\right) \\ 0 & 0 & 1 & \frac{x}{2\gamma_i} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

The appearance of the eigenvalue γ_i as a denominator element of (9) would suggest that we assume no zero eigenvalues. This is equivalent to assuming $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are nonsingular—a realistic assumption. (This was implicitly assumed in [1].) $\boldsymbol{S}_{ij}(x, \gamma_i)$ can also be determined for $\gamma_i=0$. Note that $\boldsymbol{S}(x, \gamma)$ is nonsingular for all x. Also note that for diagonal $\boldsymbol{\gamma}^2$, $\boldsymbol{S}(x, \gamma)=\boldsymbol{I}_n$, and it may easily be shown that $\boldsymbol{S}(0, \gamma)=\boldsymbol{I}_n$ and $\dot{\boldsymbol{S}}(0, \gamma) = -\dot{\boldsymbol{S}}(0, -\gamma)$.

Then the solutions for $\boldsymbol{I}$ and $\boldsymbol{V}$ are obtained as

$$\boldsymbol{I} = \boldsymbol{T}\boldsymbol{I}_m = \boldsymbol{T}\left[\boldsymbol{S}(x, \gamma)\boldsymbol{e}^{\boldsymbol{\gamma}x}\boldsymbol{a}^{+} + \boldsymbol{S}(x, -\gamma)\boldsymbol{e}^{-\boldsymbol{\gamma}x}\boldsymbol{a}^{-}\right] \tag{10a}$$

and from (2b),

$$\begin{aligned} \boldsymbol{V} &= -\boldsymbol{Y}^{-1}\dot{\boldsymbol{I}} \\ &= -\boldsymbol{Y}^{-1}\boldsymbol{T}\left[\dot{\boldsymbol{S}}(x, \gamma)\boldsymbol{e}^{\boldsymbol{\gamma}x}\boldsymbol{a}^{+} + \boldsymbol{S}(x, \gamma)\boldsymbol{\gamma}\boldsymbol{e}^{\boldsymbol{\gamma}x}\boldsymbol{a}^{+}\right. \\ &\quad \left. + \dot{\boldsymbol{S}}(x, -\gamma)\boldsymbol{e}^{-\boldsymbol{\gamma}x}\boldsymbol{a}^{-} - \boldsymbol{S}(x, -\gamma)\boldsymbol{\gamma}\boldsymbol{e}^{-\boldsymbol{\gamma}x}\boldsymbol{a}^{-}\right] \end{aligned} \tag{10b}$$

where $\boldsymbol{\gamma}$ is an $n\times n$ diagonal matrix with eigenvalues γ_i on the main diagonal in the same sequence as in $\boldsymbol{e}^{\boldsymbol{\gamma}x}$.

Evaluating (10a) and (10b) at $x=0$ yields

$$\begin{aligned} \boldsymbol{I}(0) &= \boldsymbol{T}\left[\boldsymbol{S}(0, \gamma)\boldsymbol{a}^{+} + \boldsymbol{S}(0, -\gamma)\boldsymbol{a}^{-}\right] \\ &= \boldsymbol{T}\left[\boldsymbol{a}^{+} + \boldsymbol{a}^{-}\right] \end{aligned} \tag{11a}$$

$$\begin{aligned} \boldsymbol{V}(0) &= -\boldsymbol{Y}^{-1}\boldsymbol{T}\left[\dot{\boldsymbol{S}}(0, \gamma)\boldsymbol{a}^{+} + \boldsymbol{S}(0, \gamma)\boldsymbol{\gamma}\boldsymbol{a}^{+}\right. \\ &\quad \left. + \dot{\boldsymbol{S}}(0, -\gamma)\boldsymbol{a}^{-} - \boldsymbol{S}(0, -\gamma)\boldsymbol{\gamma}\boldsymbol{a}^{-}\right] \\ &= -\boldsymbol{Y}^{-1}\boldsymbol{T}\left[(\dot{\boldsymbol{S}}(0, \gamma) + \boldsymbol{\gamma})\boldsymbol{a}^{+} + (\dot{\boldsymbol{S}}(0, -\gamma) - \boldsymbol{\gamma})\boldsymbol{a}^{-}\right] \\ &= -\boldsymbol{Y}^{-1}\boldsymbol{T}(\dot{\boldsymbol{S}}(0, \gamma) + \boldsymbol{\gamma})\left[\boldsymbol{a}^{+} - \boldsymbol{a}^{-}\right]. \end{aligned} \tag{11b}$$

Solving for $\boldsymbol{a}^{+}$ and $\boldsymbol{a}^{-}$ yields

$$\boldsymbol{a}^{+} = \tfrac{1}{2}\boldsymbol{T}^{-1}\boldsymbol{I}(0) - \tfrac{1}{2}\left[\dot{\boldsymbol{S}}(0, \gamma) + \boldsymbol{\gamma}\right]^{-1}\boldsymbol{T}^{-1}\boldsymbol{Y}\boldsymbol{V}(0) \tag{12a}$$

$$\boldsymbol{a}^{-} = \tfrac{1}{2}\boldsymbol{T}^{-1}\boldsymbol{I}(0) + \tfrac{1}{2}\left[\dot{\boldsymbol{S}}(0, \gamma) + \boldsymbol{\gamma}\right]^{-1}\boldsymbol{T}^{-1}\boldsymbol{Y}\boldsymbol{V}(0). \tag{12b}$$

From (10a), (10b), (12a), and (12b) we may obtain the matrix chain parameters of a section of line as

$$\begin{bmatrix} \boldsymbol{V}(x) \\ \boldsymbol{I}(x) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{11}(x) & \boldsymbol{\Phi}_{12}(x) \\ \boldsymbol{\Phi}_{21}(x) & \boldsymbol{\Phi}_{22}(x) \end{bmatrix}\begin{bmatrix} \boldsymbol{V}(0) \\ \boldsymbol{I}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}(x) \end{bmatrix}\begin{bmatrix} \boldsymbol{V}(0) \\ \boldsymbol{I}(0) \end{bmatrix} \tag{13}$$

where each submatrix $\boldsymbol{\Phi}_{ij}(x)$ is $n\times n$. Note that (2) appears analogous to the state-variable formulation generally written as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{14}$$

for a real-valued $\boldsymbol{A}$ matrix. Here we interpret $\boldsymbol{\Phi}(x)$ as the state-transition matrix for a complex-valued $\boldsymbol{A}$ matrix and t in (14) is analogous to x in (2) [11], [4].

A considerable simplification results when $r(\gamma_i^2\boldsymbol{I}_n - \boldsymbol{\Gamma}) = n - m_i$, for all $i=1, \cdots, k$. In this case, $\boldsymbol{\gamma}^2$ will be diagonal, $\boldsymbol{S}(x, \gamma) = \boldsymbol{S}(x, -\gamma) = \boldsymbol{I}_n$, and $\dot{\boldsymbol{S}}(x, \gamma) = {}_n\boldsymbol{0}_n$, where ${}_p\boldsymbol{0}_r$ denotes the $p\times r$ zero matrix. Then the matrix chain parameters become

$$\boldsymbol{\Phi}_{11}(x) = \tfrac{1}{2}\boldsymbol{Y}^{-1}\boldsymbol{T}(\boldsymbol{e}^{\boldsymbol{\gamma}x} + \boldsymbol{e}^{-\boldsymbol{\gamma}x})\boldsymbol{T}^{-1}\boldsymbol{Y} = \boldsymbol{Y}^{-1}\boldsymbol{T}\cosh(\boldsymbol{\gamma}x)\boldsymbol{T}^{-1}\boldsymbol{Y} \tag{15a}$$

$$\boldsymbol{\Phi}_{12}(x) = -\tfrac{1}{2}\boldsymbol{Y}^{-1}\boldsymbol{T}\boldsymbol{\gamma}(\boldsymbol{e}^{\boldsymbol{\gamma}x} - \boldsymbol{e}^{-\boldsymbol{\gamma}x})\boldsymbol{T}^{-1} = -\boldsymbol{Y}^{-1}\boldsymbol{T}\boldsymbol{\gamma}\sinh(\boldsymbol{\gamma}x)\boldsymbol{T}^{-1} \tag{15b}$$

$$\boldsymbol{\Phi}_{21}(x) = -\tfrac{1}{2}\boldsymbol{T}(\boldsymbol{e}^{\boldsymbol{\gamma}x} - \boldsymbol{e}^{-\boldsymbol{\gamma}x})\boldsymbol{\gamma}^{-1}\boldsymbol{T}^{-1}\boldsymbol{Y} = -\boldsymbol{T}\sinh(\boldsymbol{\gamma}x)\boldsymbol{\gamma}^{-1}\boldsymbol{T}^{-1}\boldsymbol{Y} \tag{15c}$$

$$\boldsymbol{\Phi}_{22}(x) = \tfrac{1}{2}\boldsymbol{T}(\boldsymbol{e}^{\boldsymbol{\gamma}x} + \boldsymbol{e}^{-\boldsymbol{\gamma}x})\boldsymbol{T}^{-1} = \boldsymbol{T}\cosh(\boldsymbol{\gamma}x)\boldsymbol{T}^{-1} \tag{15d}$$

which agrees with other results [6], [9].

The authors in [1] appear to assume that $r(\gamma_i^2 I_n - \Gamma) = n - m_i$ for all $i = 1, \cdots, k$. A correct development for distinct eigenvalues is given, i.e., $k = n$ (although they only show the solution for a line of infinite length). They appear to assume that the fully degenerate case ($k = 1$) is given by Amemiya in [7]. However, Amemiya assumes $\boldsymbol{R} = \boldsymbol{G} = {}_n0_n$ and also that $\boldsymbol{CL} = (1/v_0^2) I_n$. This therefore insures that although all eigenvalues are the same, $(\gamma_i^2 = 1/v_0^2)$, $r[\gamma_i^2 I_n - p^2 \boldsymbol{CL}] = 0$ and Γ is already in diagonal Jordan form. This would correspond to n lossless conductors imbedded in a lossless homogeneous medium [8]. One of the main results of [1, eq. (10)] seems to be in error. The modal matrix in [1, eq. (10)], $[\alpha]$ (whose columns obviously must be eigenvectors of $\boldsymbol{ZY}$), is shown for $s = n - m$ equal eigenvalues with the remaining m eigenvalues distinct.

Since [1, eq. (7)] must be a solution to [1, eq. (1c)], it seems clear that the structure of $[\alpha]$ given in [1, eq. (10)] is only valid if $\boldsymbol{ZY}$ is partially diagonal (since $[V_{ii}]$ in [1] is diagonal) as

$$\boldsymbol{ZY} = \left[\begin{array}{c|c} & {}_m0_s \\ \boldsymbol{K} & \text{----} \\ & \boldsymbol{K}^* \end{array}\right] \tag{16}$$

where $\boldsymbol{K}$ is $n \times m$ and $\boldsymbol{K}^*$ is an $s \times s$ diagonal matrix with identical scalar elements γ^{2*} on the main diagonal. This, of course, is a special case of all possible structures of $\boldsymbol{ZY}$.

There do exist certain practical cases where diagonalization of Γ can be shown *a priori*. If we neglect loss, i.e., $\boldsymbol{G} = \boldsymbol{R} = {}_n0_n$, and assume $\boldsymbol{C}$ symmetric and positive definite and $\boldsymbol{L}$ symmetric, then one may determine a real nonsingular transformation matrix $\boldsymbol{T}$ such that $\boldsymbol{T}^{-1}\Gamma\boldsymbol{T} = p^2\boldsymbol{T}^{-1}\boldsymbol{CLT}$ is diagonal. This is an application of the simultaneous diagonalization of two quadratic forms [10]. The assumption of $\boldsymbol{C}$ and $\boldsymbol{L}$ being symmetric is in most cases quite acceptable and C will be positive definite if (1) is written so that

$$[C_{ii}] = C_{iG} + \sum_{\substack{j=1 \\ j \neq i}}^{n} C_{ij} \quad \text{and} \quad [C_{ij}] = -C_{ij} \; j \neq i$$

where we denote the element of $\boldsymbol{C}$ in the ith row and jth column by $[C_{ij}]$, C_{iG} is the capacitance of the ith conductor to ground, and C_{ij} is the mutual capacitance between conductor i and conductor j. Subroutine NROOT in the IBM scientific subroutine package performs this type of reduction. Then the change of variables $I(x) = \boldsymbol{T}I_m(x)$ will "decouple" (3) with $\boldsymbol{T}^{-1} = \boldsymbol{T}^T\boldsymbol{C}^{-1}$, where we denote the transpose of a matrix $\boldsymbol{T}$ by $\boldsymbol{T}^T$.

It is also possible to include losses if we assume $\boldsymbol{R} = r(p)\boldsymbol{I}_n$ (identical conductors), $\boldsymbol{G} = {}_n0_n$, $\boldsymbol{CL} = 1/v_0^2\, I_n$, and $\boldsymbol{C}$ is symmetric, and positive definite. Then $\Gamma = pr(p)\boldsymbol{C} + p^2/v_0^2 I_n$. The transformation $I(x) = \boldsymbol{T}\boldsymbol{I}_m(x)$ such that $\boldsymbol{T}^{-1}\boldsymbol{CT}$ is diagonal will decouple (3) and the existence of $\boldsymbol{T}$ is guaranteed since $\boldsymbol{C}$ is real, symmetric.

Finally, there exist certain cyclic symmetric matrices for which diagonalization of Γ does not depend upon the entries in $\boldsymbol{Z}$ and $\boldsymbol{Y}$. For example, if $n = 3$ $[Z_{ii}] = Z$, $[Z_{12}] = [Z_{23}] = [Z_{31}] = Z'$, $[Z_{13}] = [Z_{21}] = [Z_{32}] = Z''$, and $\boldsymbol{Y}$ has a similar structure, then there exists a simple coordinate transformation $\boldsymbol{T}$ with $[T_{1j}] = [T_{ii}] = 1/\sqrt{3}$ for $i, j = 1, 2, 3$ $[T_{22}] = [T_{33}] = a^2/\sqrt{3}$, and $[T_{23}] = [T_{32}] = a/\sqrt{3}$ with $a = e^{j2\pi/3}$ and $\boldsymbol{T}^{-1} = \boldsymbol{T}^*$, where * denotes complex-conjugate transpose which will diagonalize Γ. This is sometimes referred to as a symmetrical coordinate transformation and can be extended for $n > 3$ [12]. This technique would apply to n wires within a conducting cylinder arranged symmetrically about the axis.

If $r(\gamma_i^2 I_n - \Gamma) = n - m_i$, for all $i = 1, \cdots, k$, then a set of n linearly independent eigenvectors $\boldsymbol{T}_{ij}$, for $j = 1, \cdots, m_i$, may be found satisfying

$$(\gamma_i^2 I_n - \Gamma)\boldsymbol{T}_{ij} = {}_n0_1 \tag{17}$$

where $\boldsymbol{T}_{ij}$ is an $n \times 1$ vector function of p which is also a column of $\boldsymbol{T}$. If $r(\gamma_i^2 I_n - \boldsymbol{T}) > n - m_i$ for some repeated root γ_i^2, then one may find generalized eigenvectors which place Γ in Jordan canonical form [4]. For sinusoidal excitations ($p = j\omega$), machine computation of eigenvectors is straightforward, although tedious if all n eigenvalues of Γ are not distinct.

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On the Surface-to-Bulk Mode Conversion of Rayleigh Waves

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***Abstract*—The surface-to-bulk wave conversion phenomena occurring at a discontinuity characterized by a surface contour deformation may be used as a means for tapping Rayleigh waves in a nonpiezoelectric solid. For this purpose, the mode conversion problem is treated in this short paper with the use of a boundary perturbation technique. A systematic procedure is obtained to calculate not only the first-order scattered waves which include the reflected surface wave and the converted bulk wave, but also the higher order terms. With careful design of the surface contour, the converted bulk-wave power and the direction of propagation into the substrate may be controlled.**

I. Introduction

Surface acoustic waves have received considerable attention in recent years [1], [2]. One important application of surface waves is in signal processing devices [3]–[5], where their use can reduce device length by several orders of magnitude compared with their electromagnetic counterparts. Hence, an integration of acoustic devices with integrated electronics is promising. One important factor which has put these devices into practical use is the introduction of interdigital transducers [6], [7] which have high efficiency in exciting, receiving, and tapping acoustic surface waves. However, interdigital transducers only operate on the surface of a piezoelectric crystal. For devices requiring longer delay length and more taps, larger crystals are needed, which are difficult to grow.

If a nonpiezoelectric solid is used for the main delay path in connection with a piezoelectric substrate [8]–[10], there will be no length problem, but acoustic surface waves should be tapped along the main path by methods other than interdigital fingers. In a previous paper [11], we have studied a discontinuity problem in the hope that we may use the surface-to-bulk wave transduction at a guiding discontinuity for tapping Love waves. In this short paper, we treat a similar problem for the case of Rayleigh waves. The geometry is shown in Fig. 1. For $z \leq 0$, it is a semi-infinite nonpiezoeelectric elastic medium with density ρ and Lame's elastic constants λ_e and μ, where λ is reserved to denote the Rayleigh wavelength. The guiding surface $z = 0$ has a region of deformation around $x = 0$ shown by the dotted line, while the solid line indicates a perfect surface. Consider a Rayleigh wave incident from left to right along the x axis. A bulk wave will be generated due to the discontinuity. It propagates into the substrate with certain directional characteristics which depend on the exact geometrical shape of the deformation. This bulk wave may be detected in the bottom of the substrate if the directional property of the beam is known. By the use of the boundary perturbation technique [12], [13], the mode conversion problem is systematically analyzed.

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